TRINITY COLLEGE

Year 11 Mathematics Specialist Units 1, 2
Test 5 2020

Section 1 Calculator Free

Matrices

STUDENT'S NAME Solutions

DATE: Wednesday 19 August **TIME:** 20 minutes **MARKS:** 20

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (5 marks)

Consider the system of equations $\begin{aligned} x+y&=3 \\ 2x+3y&=8 \end{aligned}$

(a) Write this in the form $AX = B$ where $X = \begin{bmatrix} x \\ y \end{bmatrix}$. [2]

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

(b) Using matrix methods, solve $AX = B$ and solve for x and y. [3]

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$= \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8 \\ -6+8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \therefore \begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

2. (8 marks)

Given the matrices A, B, C and D shown below, where possible, evaluate each of the following. If the expression cannot be evaluated, clearly explain why this is the case.

$$A=\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix},\quad B=\begin{bmatrix}6\\2\\0\end{bmatrix},\quad C=\begin{bmatrix}5 & -3\\k & -1\end{bmatrix}\quad \text{and} \quad D=\begin{bmatrix}4 & 0 & 3\end{bmatrix}$$

(a) $A+2C$ [2]

$$= \begin{bmatrix}2 & 3\\0 & 1\end{bmatrix} + 2\begin{bmatrix}5 & -3\\k & -1\end{bmatrix}$$

$$= \begin{bmatrix}12 & -3\\2k & -1\end{bmatrix}$$

(b) DB [2]

$$= \begin{bmatrix}4 & 0 & 3\end{bmatrix}\begin{bmatrix}6\\2\\0\end{bmatrix}$$

$$= \begin{bmatrix}24\end{bmatrix}$$

(c) CA [2]

$$= \begin{bmatrix}5 & -3\\k & -1\end{bmatrix}\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}10 & 12\\2k & 3k-1\end{bmatrix}$$

(d) The value of k such that CA is singular. [2]

$\det(CA) = 0$

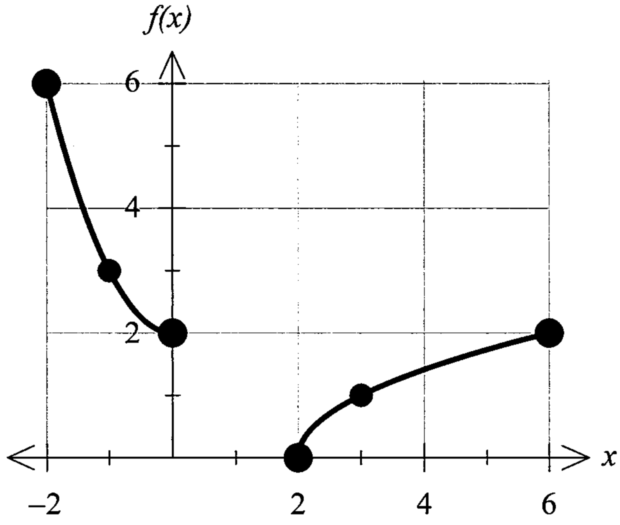
$\Rightarrow \quad 10(3k-1) - 12(2k) = 0$

$6k - 10 = 0$

$k = \frac{5}{3}$

3. (7 marks)

The points $(-2,6)$, $(-1,3)$ and $(0,2)$ on the graph of the function $f(x)=x^2+2$ are moved by a linear transformation as shown in the diagram below:



(a) State the appropriate transformation matrix. [1]

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$ Rotation 90° clockwise about origin

(b) Under a second transformation, $(2,0)$ and $(3,1)$ become $(2,0)$ and $(5,1)$ respectively.

(i) Determine the matrix that will achieve this. [2]

$$T\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} \Rightarrow T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

(ii) State the new coordinates of $(6,2)$. [1]

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix} \quad \therefore (10, 2)$$

(c) Determine the transformation matrix which will transform the points from their final positions in part (b) back to their original positions. [3]

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$$ (Combined transformation)

$$\therefore \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

TRINITY COLLEGE

Year 11 Mathematics Specialist Units 1, 2
Test 5 2020

Section 2 Calculator Assumed
Matrices

STUDENT'S NAME ______________________________

DATE: Wednesday 19 March **TIME:** 30 minutes **MARKS:** 30

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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4. (4 marks)

Solve the equation $X\begin{bmatrix} 2 & -2 \\ -7 & 4 \end{bmatrix} + X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$ for the 2×2 matrix X

$$\Rightarrow \quad X\left(\begin{bmatrix} 2 & -2 \\ -7 & 4 \end{bmatrix} + I\right) = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$\Rightarrow \quad X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 19 & 8 \\ -8 & -3 \end{bmatrix}$$

5. (10 marks)

The triangle OAB is defined by the points $O(0,0)$, $A(5,0)$ and $B(4,3)$.

(a) Determine the coordinates of the points O', A', B' of the triangle when it is transformed by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Describe this transformation geometrically. [3]

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 5 & 4 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -5 & -4 \\ 0 & 0 & -3 \end{bmatrix}$$

$O'(0,0)$ $\quad$ $A'(-5,0)$ $\quad$ $B'(-4,-3)$

$180°$ Rotation about the origin

(b) The triangle $O'A'B'$ from part (a) is then transformed by a second matrix that represents a reflection about the line $y=-x$. Write the combined effect of the two transformations as a single matrix. [3]

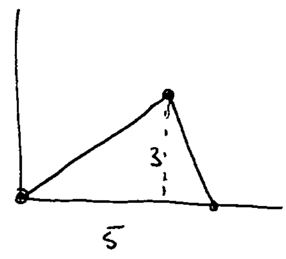
$$T_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Combined transformation

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(c) The original triangle OAB is transformed by a dilation factor 3 parallel to the x-axis and a factor k , $k > 0$, parallel to the y-axis. If the resulting image has an area of 50 square units, determine the value of k . [3]

Original area $= \frac{15}{2}$ units^2



$$T = \begin{bmatrix} 3 & 0 \\ 0 & k \end{bmatrix}$$

$$\det(T) = 3k$$

Now final area = orig area $\times |\det T|$

$$50 = \frac{15}{2} \times |3k|$$

$$k = \frac{20}{9}$$

6. (8 marks)

For any matrix M its transpose M^T is obtained by interchanging the rows and columns of M. For example

$$\text{if} \quad M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{then} \quad M^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

A matrix M is called orthogonal if $M^{-1} = M^T$.

(a) For each of the matrices that follow, write down its transpose and its inverse and, hence, decide is the matrix is orthogonal.

(i) $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ [3]

$$A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\therefore$ A is orthogonal

(ii) $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ [3]

$$B^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$\therefore$ B is <u>not</u> orthogonal

(b) In part (a) you were asked to use a method involving inspection to decide whether a square matrix is orthogonal. Justify and describe another method that could be used to determine whether a square matrix is orthogonal. This method should involve matrix multiplication. [2]

We know $MM^{-1} = I$ and that for a matrix to be orthogonal $M^{-1} = M^T$

$\therefore \quad MM^T = I \quad$ or $\quad M^TM = I$

7. (8 marks)

(a) The line with equation $y = ax + b$ is mapped to $y = -2x + 5$ after it is transformed by the linear transformation matrix $T = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ T. Determine a and b. [4]

Method 1 - use pts

$x = 0$: $(0, b) \to (0, 5)$

$x = 1$: $(1, a+b) \to (1, 3)$

So $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ b & a+b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 1 \\ 2b & 1+2a+2b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$

Solving components

$a = -3/2$

$b = 5/2$

Method 2 - inverse

$T^{-1}\begin{bmatrix} t \\ -2t+5 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \begin{cases} x = t \\ y = -\frac{1}{2}(2t-5) - \frac{1}{2}t \end{cases}$

Solving for y, t

$x = t$

$y = -0.5(3x - 5)$

$= -\frac{3}{2}x + \frac{5}{2}$

(b) Determine matrix A if $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ [4]

$\Rightarrow A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$